Component Mode Synthesis for Model Order Reduction of Nonclassically Damped Systems

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Based on a component mode synthesis approach, a model-order-reduction method for linear structures with arbitrary linear damping has been developed. Projection matrices are introduced to make the method applicable to systems having rigid-body freedom. To test the method, eigenvalues of a reduced-order model of a free-free beam with nonproportional damping were compared to exact eigenvalues and to eigenvalues obtained using two other model-reduction strategies. The present model-reduction strategy proved to be decidedly superior.

Introduction

SUCCESSFUL application of feedback control to flexible structures requires that the model order be as small as possible while remaining consistent with model accuracy requirements. This paper presents a model-order-reduction strategy based on component mode synthesis (CMS), extending the well-known CMS techniques to structures having nonclassical damping. A number of CMS, or substructure coupling, methods for dynamic analysis of structures have been developed,[1] but there is very little literature on the topic of CMS applied to damped structures. The usual approach is to assume "proportional damping," or to assume that the structure is lightly damped, to use CMS to model a reduced associated undamped system, and to supplement the reduced system with assumed modal damping. However, these approaches are invalid for structures that are subjected to Coriolis forces, direct velocity feedback control forces, or other forces that lead to nonproportional damping.

In recent years, several papers on the application of CMS methods to nonclassically damped structures have appeared. In Ref. 2, Chung and Craig developed a first-order formulation leading to complex component modes and proposed a procedure for forming a type of attachment mode to represent the component modes not kept explicitly as Ritz vectors. Hale adopted real trial vectors to reduce each substructure, and developed a procedure for increasing accuracy by iteratively generating improved substructure trial state vectors.[3] Beliveau and Soucy, in Ref. 4, extended the Craig-Bampton method[5] to damped systems with symmetric substructure matrices by replacing the real fixed-interface normal modes by the corresponding complex modes. The coupling procedure presented by Hasselman and Kaplan[6] is another extrapolation of the Craig-Bampton method, which employs complex component modes. Using variational principles, Howsman and Craig[7,8] developed a consistent state-space formulation that employs a truncated set of free-interface component modes along with a set of attachment modes.

This paper presents a new free-interface method of component mode synthesis for structures with rigid-body freedom and with arbitrary linear damping. A complex mode equiva-

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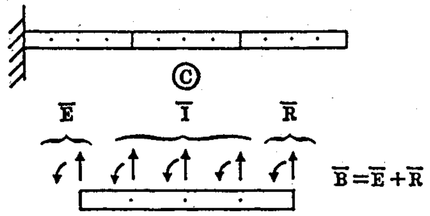

Fig. 1 A beam-including component with redundant boundary.

lent of residual flexibility is first developed, and then state-variable residual inertia-relief attachment modes are defined and are employed, along with a set of complex modes (eigenvectors), in a component coupling procedure. The results of an example problem demonstrate that the component synthesis approach presented here leads to far greater accuracy of eigenvalues of the reduced system than heretofore achieved by previous methods.

Component Equations in State-Variable Form

Figure 1 shows a beam divided into substructures, or components. The interface, or boundary, coordinates that a component shares with adjacent components are labeled B, while the remaining coordinates are labeled I (interior). The boundary coordinates may be further subdivided into R (rigid-body) and E (excess or redundant) subsets.

The equation of motion for a typical free-interface component of a damped structure may be written

$$m\ddot{x} + c\dot{x} + kx = f \tag{1}$$

where m, c, and k are the $(n \times n)$ mass, damping, and stiffness matrices, respectively. There is no assumption that the matrices in Eq. (1) are symmetric, although m and k will normally be symmetric. However, if the component has rigid-body freedom, k will be singular.

Where necessary, Eq. (1) will be expanded into (i,b) partitions or (i,e,r) partitions in accordance with the notation of Fig. 1. In this paper it will be assumed that there are no external forces acting on the structure, so the only forces exerted on a component act on the boundary, and f has the form

$$f = \begin{bmatrix} 0_i \\ f_b \end{bmatrix} \tag{2}$$

As in Refs. 7–9, external forces could be added in a straightforward manner.

Equation (1) can be expanded to $2n$-order state-space form as follows:

$$A\dot{X} + BX = F \tag{3}$$

where

$$A = \begin{bmatrix} 0 & m \\ m & c \end{bmatrix}, B = \begin{bmatrix} -m & 0 \\ 0 & k \end{bmatrix}, X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, F = \begin{bmatrix} 0 \\ f \end{bmatrix} \tag{4}$$

A and B will be referred to as the state mass matrix and the state stiffness matrix, respectively, and X will be called the state displacement vector.

Using a variational approach, Howsman and Craig[7,8] have shown that, corresponding to Eq. (3), there is an adjoint differential equation

$$-A^T\dot{Y} + B^TY = F^* \tag{5}$$

where the adjoint state displacement vector Y and adjoint state force vector F^* are given by

$$Y = \begin{bmatrix} \dot{y} \\ y \end{bmatrix}, F^* = \begin{bmatrix} 0 \\ f^* \end{bmatrix} \tag{6}$$

If m, c, and k are all symmetric, the system is self-adjoint, and Eq. (5) is not needed. The adjoint equation plays a significant role in the theory developed in this paper.

Let λ, ϕ, and ψ be an eigenvalue, right eigenvector, and left eigenvector, respectively. Then, the right eigenproblem is obtained by substituting $X = \phi e^{\lambda t}$ into Eq. (3) (with $F = 0$), and the left eigenproblem is obtained by substituting $Y = \psi e^{-\lambda t}$ into Eq. (5) (with $F^* = 0$). Thus,

$$(\lambda_i A + B)\phi_i = 0 \qquad i = 1,2,\ldots,2n \tag{7}$$

and

$$(\lambda_i A^T + B^T)\psi_i = 0 \qquad i = 1,2,\ldots,2n \tag{8}$$

Let the complex spectrum matrix be denoted by Λ. (Λ will be diagonal, with the $2n$ eigenvalues λ_i on the principal diagonal. Exceptions in which the eigensystem is defective and Λ has Jordan form include systems with rigid-body modes. Such cases will be discussed in a subsequent paper. In the remainder of this paper, with the exception of the example problem, it will be assumed that Λ is diagonal, i.e., the system is nondefective.) Further, let the right complex mode matrix, whose columns are the right eigenvectors, be denoted by Φ, and the left complex mode matrix, whose columns are the left eigenvectors, be denoted by Ψ. Then Eqs. (7) and (8) may be written in the form

$$B\Phi = -A\Phi\Lambda \tag{9}$$

$$\Psi^T B = -\Lambda\Psi A \tag{10}$$

Premultiplication of Eq. (9) by Ψ^T and postmultiplication of Eq. (10) by Φ yields

$$\Psi^T B\Phi = -\Psi^T A\Phi\Lambda = -\Lambda\Psi^T A\Phi \tag{11}$$

or

$$\bar{B} = -\bar{A}\Lambda = -\Lambda\bar{A} \tag{12}$$

where

$$\bar{A} = \Psi^T A\Phi, \qquad \bar{B} = \Psi^T B\Phi \tag{13}$$

Except for rows and columns that correspond to repeated roots λ_i, both $\bar{A}$ and $\bar{B}$ will be diagonal due to the following biorthogonality relationships:

$$\psi_i^T A\phi_j = 0, \qquad \psi_i^T B\phi_j = 0 \qquad \lambda_i \neq \lambda_j \tag{14}$$

Projection Matrix for Undamped Systems

To set the stage for the development of reduced-order component equations for damped structures, several key concepts from CMS theory for undamped components will be reviewed briefly. More details are given in Refs. 1 and 10.

For structures with rigid-body freedom, a projection matrix P can be defined that has the following properties:

1) When a force vector f is premultiplied by the projection matrix P, a selfequilibrated force vector is created, whereby f is equilibrated by rigid-body inertia forces.

2) When a displacement vector x is premultiplied by P^T, the rigid-body components of motion are removed from x leaving only the flexible components of x.

The projection matrix for undamped systems is given by

$$P = I - m\Phi_r\mu_{rr}^{-1}\Phi_r^T \tag{15}$$

where Φ_r is the $(n \times n_r)$ matrix of rigid-body modes, μ_{rr}^{-1} the inverse of $\Phi_r^T m\Phi_r$, and n_r the number of rigid-body freedoms of the structure.

The projection matrix for undamped systems is employed in defining an elastic flexibility matrix and in defining inertia-relief attachment modes. For structures with rigid-body freedom, the stiffness matrix k is singular and has the rank $(n - n_r)$. Therefore, k cannot be inverted to obtain a flexibility matrix. However, an elastic flexibility matrix (pseudoinverse of k) can be defined as

$$G_f = \Phi_f \Lambda_{ff}^{-1} \Phi_f^T = P^T G P \tag{16}$$

where Φ_f is the $n \times (n - n_r)$ matrix of flexible undamped modes, and Λ_{ff} the $(n - n_r) \times (n - n_r)$ diagonal matrix of nonzero eigenvalues. The flexibility matrix G is obtained by inverting the stiffness matrix for the structure when it is supported at n_r freedoms.

The projection matrix P and elastic flexibility matrix G_f are employed in the procedure for defining so-called inertia-relief attachment modes. These modes were originally introduced by MacNeal[11] and were later used by Rubin[12] and by Craig and Chang.[13] Let f_a be defined by the $n \times n_b$ matrix

$$f_a = \begin{bmatrix} 0_{ib} \\ I_{bb} \end{bmatrix} \tag{17}$$

Then, a set of n_b inertia-relief attachment modes based on unit forces at the boundary freedoms may be defined by

$$\Phi_a = G_f f_a \tag{18}$$

In this paper, the concept of the projection matrix is extended to structures with general linear damping and is employed in defining inertia-relief attachment modes for damped structures.

Projection Matrix for Damped Substructures

In order to develop substructure (component) Ritz vectors for damped systems in a manner similar to that employed for undamped systems in Ref. 10, the equations of motion in state variable form will be employed. Projection matrices will be defined and then will be used in defining attachment modes for damped components.

The right complex mode matrix Φ and the left complex mode matrix Ψ may be partitioned in the following manner:

$$\Phi = [\Phi_r \Phi_f], \qquad \Psi = [\Psi_r \Psi_f] \tag{19}$$

where subscripts r and f denote the rigid-body mode partition and the flexible complex mode partition, respectively. Then Eq. (13) can be expanded to give

$$\begin{bmatrix} \Psi_r^T \\ \Psi_f^T \end{bmatrix} A [\Phi_r \; \Phi_f] = \begin{bmatrix} \bar{A}_{rr} & 0 \\ 0 & \bar{A}_{ff} \end{bmatrix} \tag{20}$$

where

$$\bar{A}_{rr} = \Psi_r^T A \Phi_r, \qquad \bar{A}_{ff} = \Psi_f^T A \Phi_f \tag{21}$$

If there are no repeated eigenvalues, $\bar{A}_{ff}$ is a diagonal matrix. However, $\bar{A}_{rr}$ will not necessarily ever be diagonal. An equation for B similar to Eq. (20) gives

$$\bar{B}_{rr} = \Psi_r^T B \Phi_r, \qquad \bar{B}_{ff} = \Psi_f^T B \Phi_f \tag{22}$$

It follows from Eq. (20) that

$$\Phi^{-1} = \begin{bmatrix} \bar{A}_{rr}^{-1} \Psi_r^T A \\ \bar{A}_{ff}^{-1} \Psi_f^T A \end{bmatrix} \tag{23}$$

and

$$A^{-1} = \Phi_r \bar{A}_{rr}^{-1} \Psi_r^T + \Phi_f \bar{A}_{ff}^{-1} \Psi_f^T \tag{24}$$

For damped systems, a right projection matrix P is defined such that if a state displacement vector X is premultiplied by P^T, the rigid-body modes will be removed from the vector. It follows from this definition that

$$P^T [\Phi_r \Phi_f] = [0 \; \Phi_f] \tag{25}$$

Postmultiplying Eq. (25) by Φ^{-1} and combining the result with Eq. (21) yields

$$P^T = [0 \; \Phi_f][\Phi_r \Phi_f]^{-1} = \Phi_f \bar{A}_{ff}^{-1} \Psi_f^T A \tag{26}$$

Finally, Eq. (26) may be substituted into Eq. (24) to obtain an alternative expression for P^T, namely

$$P^T = I - \Phi_r \bar{A}_{rr}^{-1} \Psi_r^T A \tag{27}$$

In a similar manner, a left projection matrix Q may be defined such that

$$Q^T [\Psi_r \Psi_f] = [0 \; \Psi_f] \tag{28}$$

Then Q^T will be of the following form

$$Q^T = \Psi_f \bar{A}_{ff}^{-1} \Phi_f^T A^T \tag{29}$$

or, equivalently,

$$Q = I - A \Phi_r \bar{A}_{rr}^{-1} \Psi_r^T \tag{30}$$

The complex spectrum matrix Λ in Eqs. (9) and (10) can be partitioned into rigid-body and elastic partitions as

$$\Lambda = \begin{bmatrix} \Lambda_{rr} & 0 \\ 0 & \Lambda_{ff} \end{bmatrix} = \begin{bmatrix} 0_{rr} & 0 \\ 0 & \Lambda_{ff} \end{bmatrix} \tag{31}$$

since Λ_{rr} is null. (There are special cases where Λ_{rr} is not null but, rather, has Jordan form. However, as noted earlier, this paper considers only the nondefective case, for which Λ_{rr} will be null.) Substitution of Eqs. (31) and (19) into Eqs. (9) and (10) gives

$$B [\Phi_r \Phi_f] = -A [\Phi_r \Phi_f] \begin{bmatrix} 0_{rr} & 0 \\ 0 & \Lambda_{ff} \end{bmatrix} \tag{32}$$

and

$$\begin{bmatrix} \Psi_r^T \\ \Psi_f^T \end{bmatrix} B = - \begin{bmatrix} 0_{rr} & 0 \\ 0 & \Lambda_{ff} \end{bmatrix} \begin{bmatrix} \Psi_r^T \\ \Psi_f^T \end{bmatrix} A \tag{33}$$

From these last two equations the following four equations may be deduced:

$$B\Phi_r = 0, \qquad B\Phi_f = -A \Phi_f \Lambda_{ff}$$
$$\Psi_r^T B = 0, \qquad \Psi_f^T B = -\Lambda_{ff} \Psi_f^T A \tag{34}$$

The following relationships can be shown to hold for the projection matrices P and Q and their transposes:

$$P^T P^T = P^T, \qquad PP = P$$
$$Q^T Q^T = Q^T, \qquad QQ = Q$$
$$QB = B, \qquad BP^T = B$$
$$QA = AP^T \tag{35}$$

In Eqs. (25) and (28), P^T and Q^T where defined such that they nullified rigid-body modes. Thus, if X is expressed as a superposition of right eigenvectors by

$$X = \Phi_r \eta_r + \Phi_f \eta_f \tag{36}$$

Then

$$P^T X = \Phi_f \eta_f = X - X_r \tag{37}$$

That is, premultiplication of an arbitrary state displacement vector X by P^T removes the rigid-body component of X.

If the force F in Eq. (3) is premultiplied by Q, the following result is obtained:

$$\begin{aligned} QF &= Q(A\dot{X} + BX) \\ &= QA\dot{X} + QBX = AP^T\dot{X} + BX \\ &= A(\dot{X} - \dot{X}_r) + BX = F - AX_r \\ &= \begin{bmatrix} 0 \\ f \end{bmatrix} - \begin{bmatrix} 0 & m \\ m & c \end{bmatrix} \begin{bmatrix} \ddot{x}_r \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} -m\dot{x}_r \\ f - m\ddot{x}_r - c\dot{x}_r \end{bmatrix} \end{aligned} \tag{38}$$

Thus, it is seen in the lower portion of the state-space form of QF that the applied force is equilibrated by the inertia and damping forces resulting from the rigid-body motion of the structure.

The projection matrices P and Q not will be used to define analogs of the elastic flexibility matrix, Eq. (16), and inertia-relief attachment modes, Eq. (18), for damped systems.

Inertia-Relief Attachment Modes for Damped Systems

In component mode synthesis, it is common practice (e.g., see Ref. 10) to use as Ritz vectors a subset of the component's normal modes (eigenvectors) together with some static modes that account for the flexibility of the modes that are not retained. This leads to the concepts of residual flexibility and residual attachment modes, which will now be generalized to damped systems.

In this section, the projection matrices introduced in the previous section will be employed to define state inertia-relief attachment modes and a state elastic flexibility matrix. Then state residual inertia-relief attachment modes will be defined and an explanation of the physical significance of the generalized coordinates associated with these residual modes will be given.

To define the attachment modes, first let F_a be the $(2n \times n_b)$ matrix of state forces with unit forces applied at each boundary coordinate. That is,

$$F_a = \begin{bmatrix} 0_{ib} \\ 0_{bb} \\ \cdots \\ 0_{ib} \\ I_{bb} \end{bmatrix} = \begin{bmatrix} 0_{ie} & 0_{ir} \\ 0_{ee} & 0_{er} \\ 0_{re} & 0_{rr} \\ \cdots & \cdots \\ 0_{ie} & 0_{ir} \\ I_{ee} & 0_{er} \\ O_{re} & I_{rr} \end{bmatrix} \tag{39}$$

Then, let $\hat{\Phi}_a$ be the matrix of static state displacement vectors of a component loaded by QF_a and supported on a user-defined r set of boundary freedoms that provide restraint against rigid-body motion. Let a pseudoflexibility matrix D be defined by

$$D = \begin{bmatrix} -m^{-1} & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & \multicolumn{2}{c}{\begin{bmatrix} k_{ii} & k_{ie} \\ k_{ei} & k_{ee} \end{bmatrix}^{-1}} & 0 \\ 0 & \vdots & & & 0 \\ 0 & \vdots & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

where it is assumed that m is nonsingular. Then, $\hat{\Phi}_a$ is given by

$$\hat{\Phi}_a = DQF_a \tag{41}$$

To remove the rigid-body modes from $\hat{\Phi}_a$, Eq. (41) may be premultiplied by P^T, leading to the following definition of right state inertia-relief attachment modes:

$$\Phi_a = P^T DQF_a \tag{42}$$

or

$$\Phi_a = D_f F_a \tag{43}$$

where

$$D_f = P^T DQ \tag{44}$$

D_f will be referred to as the state elastic flexibility matrix. D_f can also be expressed in the form

$$D_f = \Phi_f \bar{B}_{ff}^{-1} \Psi_f^T \tag{45}$$

In order to extend the concepts of residual flexibility and residual attachment modes to damped systems, let the elastic complex mode sets be partitioned into kept (preserved) modes and deleted (higher) modes as follows:

$$\Phi_f = [\Phi_p \; \Phi_h], \qquad \Psi_f = [\Psi_p \; \Psi_h] \tag{46}$$

Then,

$$\begin{aligned} D_f &= [\Phi_p \; \Phi_h] \begin{bmatrix} \bar{B}_{pp}^{-1} & 0 \\ 0 & \bar{B}_{hh}^{-1} \end{bmatrix} \begin{bmatrix} \Psi_p^T \\ \Psi_h^T \end{bmatrix} \\ &= D_p + D_h \end{aligned} \tag{47}$$

where

$$D_p = \Phi_p \bar{B}_{pp}^{-1} \Psi_p^T, \qquad D_h = \Phi_h \bar{B}_{hh}^{-1} \Psi_h^T \tag{48}$$

From Eqs. (44), (47), and (48), the following expression for D_h may be obtained:

$$D_h = P^T DQ - \Phi_p \bar{B}_{pp}^{-1} \Psi_p^T \tag{49}$$

D_h is referred to as the state residual elastic flexibility matrix. It may be used, together with Eq. (43), to define a right state residual inertia-relief attachment mode matrix

$$\Phi_d = D_h F_a \tag{50}$$

Φ_d can be written in a more explicit form if Φ_d, Φ_h, and Ψ_h are first written in expanded form as

$$\Phi_d = \begin{bmatrix} \Phi_{id}^V \\ \Phi_{bd}^V \\ \Phi_{id}^D \\ \Phi_{bd}^D \end{bmatrix}, \quad \Phi_h = \begin{bmatrix} \Phi_{ih}^V \\ \Phi_{bh}^V \\ \Phi_{ih}^D \\ \Phi_{bh}^D \end{bmatrix}, \quad \Psi_h = \begin{bmatrix} \Psi_{ih}^V \\ \Psi_{bh}^V \\ \Psi_{ih}^D \\ \Psi_{bh}^D \end{bmatrix} \tag{51}$$

where V and D refer to the velocity and displacement partitions of the state vectors. Then, from Eqs. (39), (48), (50), and (51),

$$\Phi_d = \Phi_h \bar{B}_{hh}^{-1} \Psi_h^T F_a = \Phi_h \bar{B}_{hh}^{-1} (\Psi_{bh}^D)^T \tag{52}$$

The displacement boundary partition of Φ_d, which will be required later, is given by

$$\Phi_{bd}^D = \Phi_{bh}^D \bar{B}_{hh}^{-1} (\Psi_{bh}^D)^T \tag{53}$$

Thus, Φ_{bd}^D is just the boundary displacement partition of the matrix D_h of Eqs. (48) and (49), namely

$$\Phi_{bd}^D = (D_h)_{bb}^{DD} \tag{54}$$

From Eqs. (7) and (8) it is noted that B^T has the same relationship to the left eigenvector that B has to the right eigenvector. By analogy, a left state residual inertia-relief attachment mode matrix Ψ_d is defined by

$$\Psi_d = D_h^T F_a \tag{55}$$

It can be shown that

$$\Psi_d = \Psi_h(\bar{B}_{hh}^{-1})^T(\Phi_{bh}^D)^T \tag{56}$$

whose boundary displacement partition is given by

$$\Psi_{bd}^D = \Psi_{bh}^D(\bar{B}_{hh}^{-1})^T(\Phi_{bh}^D)^T \tag{57}$$

By comparing Eqs. (53) and (57), it is seen that

$$\Psi_{bd}^D = (\Phi_{bd}^D)^T \tag{58}$$

If m, c, and k are all symmetric, $A, B, D, D_f, \Phi_{bd}^D$, and Ψ_{bd}^D are all symmetric. Furthermore, $\Psi = \Phi$ and $Q = P$.

The state-vector of physical coordinates X of Eq. (3) may be represented in terms of component generalized coordinates by the coordinate transformation

$$X = \Sigma U = [\Phi_k \Phi_d]\begin{bmatrix} U_k \\ U_d \end{bmatrix} \tag{59}$$

where it is now assumed that the rigid-body modes Φ_r and a subset Φ_p of the matrix of elastic complex modes are both included in Φ_k, and where Φ_d is given by Eq. (50). Let

$$\Pi = [\Psi_k \Psi_d] \tag{60}$$

where it is assumed that Ψ_k also includes rigid-body modes as well as a subset of elastic left eigenvectors, and where Ψ_d is given by Eq. (55). Then, Eq. (3) may be transformed (reduced) to generalized coordinates as follows:

$$\Pi^T A \Sigma \dot{U} + \Pi^T B \Sigma U = \Pi^T F \tag{61}$$

From Eqs. (48) and (50) it is seen that each column in Φ_d is a linear combination of the columns of Φ_h, and from Eq. (56) it is seen that each column of Ψ_d is a linear combination of the columns of Ψ_h. It follows from the biorthogonality equation, Eq. (14), that

$$\Pi^T A \Sigma = \begin{bmatrix} \bar{A}_{kk} & 0 \\ 0 & \bar{A}_{dd} \end{bmatrix}, \qquad \Pi^T B \Sigma = \begin{bmatrix} \bar{B}_{kk} & 0 \\ 0 & \bar{B}_{dd} \end{bmatrix} \tag{62}$$

where

$$\bar{A}_{kk} = \Psi_k^T A \Phi_k, \qquad \bar{A}_{dd} = \Psi_d^T A \Phi_d$$
$$\bar{B}_{kk} = \Psi_k^T B \Phi_k, \qquad \bar{B}_{dd} = \Psi_d^T B \Phi_d \tag{63}$$

It can be shown further that $\bar{A}_{dd}$ and $\bar{B}_{bd}$ have the forms

$$\bar{A}_{dd} = \Phi_{bh}^D \bar{A}_{hh}^{-1} \Lambda_{hh}^{-2} (\Psi_{bh}^D)^T$$
$$\bar{B}_{dd} = \Phi_{bd}^D \tag{64}$$

If Π in Eq. (60) is partitioned as

$$\Pi = [\Psi_k \Psi_d] = \begin{bmatrix} \Psi_{ik}^V & \Psi_{id}^V \\ \Psi_{bk}^V & \Psi_{bd}^V \\ \Psi_{ik}^D & \Psi_{id}^D \\ \Psi_{bk}^D & \Psi_{bd}^D \end{bmatrix} \tag{65}$$

and F represents forces applied only at the boundary, so that

$$F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_b \end{bmatrix} \tag{66}$$

then $\Pi^T F$ has the form

$$\Pi^T F = \begin{bmatrix} (\Psi_{bk}^D)^T \\ (\Psi_{bd}^D)^T \end{bmatrix} f_b \tag{67}$$

Now Eq. (61) may be written in the form

$$\begin{bmatrix} \bar{A}_{kk} & 0 \\ 0 & \bar{A}_{dd} \end{bmatrix}\begin{bmatrix} \dot{U}_k \\ \dot{U}_d \end{bmatrix} + \begin{bmatrix} \bar{B}_{kk} & 0 \\ 0 & \bar{B}_{dd} \end{bmatrix}\begin{bmatrix} U_k \\ U_d \end{bmatrix} = \begin{bmatrix} (\Psi_{bk}^D)^T \\ (\Psi_{bd}^D)^T \end{bmatrix} f_b \tag{68}$$

Incorporation of Eqs. (58) and (64) into the lower partition of Eq. (68) gives

$$\bar{A}_{dd}\dot{U}_d + \Phi_{bd}^D U_d = \Phi_{bd}^D f_b \tag{69}$$

If the response U_d is approximated by the pseudostatic response obtained by setting $\dot{U}_d = 0$, then Eq. (69) gives

$$\Phi_{bd}^D(U_d - f_b) = 0 \tag{70}$$

But, since Φ_{bd}^D is nonsingular, the pseudostatic approximation of U_d is seen to be just the boundary force vector, i.e.,

$$U_d = f_b \tag{71}$$

An analogous result for undamped systems was first developed by Craig and Chang in Ref. 13 in conjunction with their Ritz derivation of Rubin's method (see also Sec. 19.5 of Ref. 10).

Coupling of Damped Components

The component coupling procedure described next parallels the procedure employed for undamped systems in Refs. 1, 2, and 10. However, it also incorporates significant features first derived for damped systems by Howsman and Craig using a variational approach.[7,8]

Let the damped system under consideration be composed of two components, α and β, which have a common interface, as shown in Fig. 2.

The physical displacements at the interface are subject to a compatibility equation

$$x_b^\alpha = x_b^\beta \tag{72}$$

References 7 and 8 indicate that Eq. (72) implies an adjoint compatibility equation

$$y_b^\alpha = y_b^\beta \tag{73}$$

where y is defined in Eq. (6).

The interface forces are related by

$$f_b^\alpha + f_b^\beta = 0 \tag{74}$$

and the corresponding adjoint force equilibrium equation is

$$f_b^{*\alpha} + f_b^{*\beta} = 0 \tag{75}$$

Equation (59) and a similar adjoint equation can be written for each component. That is,

$$X^\alpha = \Sigma^\alpha U^\alpha = [\Phi_k^\alpha \ \Phi_d^\alpha]\begin{bmatrix} U_k^\alpha \\ U_d^\alpha \end{bmatrix} \tag{76a}$$

$$Y^\alpha = \Pi^\alpha V^\alpha = [\Psi_k^\alpha \ \Psi_d^\alpha]\begin{bmatrix} V_d^\alpha \\ V_d^\alpha \end{bmatrix} \tag{76b}$$

$$X^\beta = \Sigma^\beta U^\beta, \qquad Y^\beta = \Pi^\beta V^\beta \tag{76c}$$

where, again, Φ_k^α and Ψ_k^α include rigid-body modes as well as the kept complex flexible modes.

Reduced state equations of the form of Eq. (68) can be written for both components and "stacked" together to form

$$\begin{bmatrix}\bar{A}^\alpha_{kk} & 0 & 0 & 0\\ 0 & \bar{A}^\alpha_{dd} & 0 & 0\\ 0 & 0 & \bar{A}^\beta_{kk} & 0\\ 0 & 0 & 0 & \bar{A}^\beta_{dd}\end{bmatrix}\begin{bmatrix}\dot{U}^\alpha_k\\ \dot{U}^\alpha_d\\ \dot{U}^\beta_k\\ \dot{U}^\beta_d\end{bmatrix} + \begin{bmatrix}\bar{B}^\alpha_{kk} & 0 & 0 & 0\\ 0 & \bar{B}^\alpha_{dd} & 0 & 0\\ 0 & 0 & \bar{B}^\beta_{kk} & 0\\ 0 & 0 & 0 & \bar{B}^\beta_{dd}\end{bmatrix}\begin{bmatrix}U^\alpha_k\\ U^\alpha_d\\ U^\beta_k\\ U^\beta_d\end{bmatrix} = \begin{bmatrix}(\Psi^{D\alpha}_{bk})^T\\ (\Psi^{D\alpha}_{bd})^T\\ -(\Psi^{D\beta}_{bk})^T\\ -(\Psi^{D\beta}_{bd})^T\end{bmatrix} f^\alpha_b \qquad (77)$$

where Eq. (74) has ben employed to express f^β_b in terms of f^α_b. In condensed form, Eq. (77) may be written as

$$A_s \dot{U}_s + B_s U_s = R^T f^\alpha_b \qquad (78)$$

Because of the constraint equations (72–75), the coordinates U and the corresponding adjoint coordinates V are not sets of independent coordinates. Equations (72–75) may be combined with Eqs. (71), (76), and the adjoint equivalent of Eq. (71) to give the following constraint equations

$$\begin{bmatrix}\Phi^{D\alpha}_{bk} & \Phi^{D\alpha}_{bd} & -\Phi^{D\beta}_{bk} & -\Phi^{D\beta}_{bd}\\ 0 & I^\alpha_{bd} & 0 & I^\beta_{bd}\end{bmatrix}\begin{bmatrix}U^\alpha_k\\ U^\alpha_d\\ U^\beta_k\\ U^\beta_d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad (79a)$$

$$\begin{bmatrix}\Psi^{D\alpha}_{bk} & \Psi^{D\alpha}_{bd} & -\Psi^{D\beta}_{bk} & -\Psi^{D\beta}_{bd}\\ 0 & I^\alpha_{bd} & 0 & I^\beta_{bd}\end{bmatrix}\begin{bmatrix}V^\alpha_k\\ V^\alpha_d\\ V^\beta_k\\ V^\beta_d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad (79b)$$

Let

$$Z = \begin{bmatrix}U^\alpha_k\\ U^\beta_k\end{bmatrix}, \qquad W = \begin{bmatrix}V^\alpha_k\\ V^\beta_k\end{bmatrix} \qquad (80)$$

be the vectors of independent generalized coordinates. Then, from Eqs. (79) and (80), the transformations relating the dependent coordinate sets U_s and V_s to the independent coordinate sets Z and W are

$$U_s = SZ, \qquad V_s = TW \qquad (81)$$

where

$$S = \begin{bmatrix}I & 0\\ [-a_\phi \Phi^{D\alpha}_{bk}] & [a_\phi \Phi^{D\beta}_{bk}]\\ 0 & I\\ [a_\phi \Phi^{D\alpha}_{bk}] & [-a_\phi \Phi^{D\beta}_{bk}]\end{bmatrix} \qquad T = \begin{bmatrix}I & 0\\ [-a_\psi \Psi^{D\alpha}_{bk}] & [a_\psi \Psi^{D\beta}_{bk}]\\ 0 & I\\ [a_\psi \Psi^{D\alpha}_{bk}] & [-a_\psi \Psi^{D\beta}_{bk}]\end{bmatrix} \qquad (82)$$

where

$$a_\phi = [\Phi^{D\alpha}_{bd} + \Phi^{D\beta}_{bd}]^{-1}, \qquad a_\psi = [\Psi^{D\alpha}_{bd} + \Psi^{D\beta}_{bd}]^{-1} \qquad (83)$$

Equation (78) may now be condensed to a set of $(n_{k\alpha} + n_{k\beta})$ equations in Z by substituting Eq. (81) into Eq. (78) and pre-

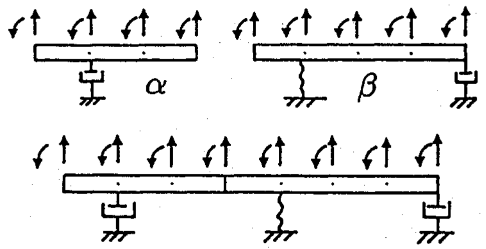

Fig. 2 Components and coupled system.

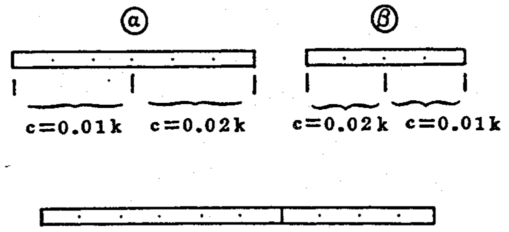

Fig. 3 Free free beam components and system.

multiplying the resulting equation by T^T. This gives the final reduced system equation of motion

$$A_z \dot{Z} + B_z Z = 0 \qquad (84)$$

where

$$A_z = T^T A_s S = \begin{bmatrix}[\bar{A}^\alpha_{kk} + (\Psi^{D\alpha}_{bk})^T a \Phi^{D\alpha}_{bk}] & [-(\Psi^{D\alpha}_{bk})^T a \Phi^{D\beta}_{bk}]\\ [-(\Psi^{D\beta}_{bk})^T a \Phi^{D\alpha}_{bk}] & [\bar{A}^\beta_{kk} + (\Psi^{D\beta}_{bk})^T a \Phi^{D\beta}_{bk}]\end{bmatrix} \qquad (85)$$

and

$$B_z = T^T B_s S = \begin{bmatrix}[\bar{B}^\alpha_{kk} + (\Psi^{D\alpha}_{bk})^T b \Phi^{D\alpha}_{bk}] & [-(\Psi^{D\alpha}_{bk})^T b \Phi^{D\beta}_{bk}]\\ [-(\Psi^{D\beta}_{bk})^T b \Phi^{D\alpha}_{bk}] & [\bar{B}^\beta_{kk} + (\Psi^{D\beta}_{bk})^T b \Phi^{D\beta}_{bk}]\end{bmatrix} \qquad (86)$$

where

$$a = a^T_\psi [\bar{A}^\alpha_{dd} + \bar{A}^\beta_{dd}] a_\phi$$
$$b = a^T_\psi [\bar{B}^\alpha_{dd} + \bar{B}^\beta_{dd}] a_\phi \qquad (87)$$

The term on the right-hand side of Eq. (78) vanishes when the equation is premultiplied by T^T since $T^T R^T = 0$.

Equation (84) is the final reduced system equation of motion. The original displacements can be recovered from U_s by using the transformation of Eqs. (76) and (81). Thus,

$$X_s \equiv \begin{bmatrix}X_\alpha\\ X_\beta\end{bmatrix} = T_s Z \qquad (88)$$

where

$$T_s = \begin{bmatrix}[\Phi^{D\alpha}_k - \Phi^{D\alpha}_d a_\phi \Phi^{D\alpha}_{bk}] & [\Phi^{D\alpha}_d a_\phi \Phi^{D\beta}_{bk}]\\ [\Phi^{D\beta}_d a_\phi \Phi^{D\alpha}_{bk}] & [\Phi^{D\beta}_k - \Phi^{D\beta}_d a_\phi \Phi^{D\beta}_{bk}]\end{bmatrix} \qquad (89)$$

Example Problem

In order to compare the current CMS model-reduction method with those proposed in Ref. 8, example 2 of Ref. 8 was chosen as a test problem. This example is a free-free beam with 22 physical degrees of freedom (DOF) (44 elements in the state vector). The beam is divided into two components, α and β, as shown, and the damping is taken to be proportional to the element stiffness as indicated on Fig. 3. Note that the damping is proportional to the stiffness at the element level, but it is not proportional to stiffness either at the component level or the system level.

The method of the present paper differs from Ref. 8 both with regard to the definition of attachment modes and to the compatibility equations used in the coupling analysis. Three cases are presented. In cases 1 and 2, the attachment modes defined in Ref. 8 are employed, while case 3 uses the attachment modes defined in this paper. The component modes are summarized in Table 1.

In cases 1 and 2, both displacement and velocity compatibility at the interface were enforced, resulting in a reduced eigenproblem of order 28 for these two cases. In case 3, the physical displacement compatibility at the interface and the constraint equation based on Eqs. (71) and (74) were enforced, which also gives a 28-DOF reduced system model.

Table 2a presents a comparison of the modal damping coefficients, σ_i, of the three cases with the exact damping coefficients of the 44-DOF original problem, and Table 2b provides a comparison of the damped natural frequencies, ω_{d_i}, where σ_i and ω_{d_i} are the real and imaginary parts of the complex eigenvalues, i.e.,

$$\lambda_i = \sigma_i + j\omega_{d_i} \tag{90}$$

Eigenvalues of the rigid-body modes are not included in the tables.

It can be seen from Tables 2a and 2b that both the real part and the imaginary part of the eigenvalues of the reduced-order model created by the procedure presented in this paper (case 3) are, in general, an order of magnitude more accurate than the eigenvalues obtained in Ref. 8 (cases 1 and 2).

Table 1 Listing of component modes used in test cases

DOF α	DOF β	Case 1	Case 2	Case 3
4	4		Rigid-body modes[a]	
6×2	4×2		Pairs of elastic complex modes	
2	2	Generalized inertia-relief attachment modes[8]	Restrained substructure attachment modes[8]	Residual inertia-relief attachment modes [Eq. (50)]
18	14	⇐TOTAL		

[a]In this example problem each component has a pair of regular rigid-body modes and a pair of corresponding generalized rigid-body modes since the damping is proportional to stiffness on an element-by-element basis. Cases such as this, where generalized eigenvectors and the Jordan form of Λ are required, are discussed at greater length in a subsequent paper.[14]

Computational Considerations

As noted in Eqs. (49), (50), and (55), both left and right complex eigenvectors of the components are required in forming the state residual inertia-relief attachment modes. These enter into the formation of the final reduced system matrices A_z and B_z in Eq. (84), which are complex matrices. Therefore, it is necessary that a complex eigensolver be employed.

However, it should be pointed out that not all matrix computations need be carried out in complex form. The state-vector rigid-body modes Φ_r and Ψ_r are real. Thus, matrices such as $\bar{A}_{rr}$, P, Q, D, and D_f are all real matrices. Furthermore, when k and m are symmetric (as is generally the case) Φ_r and Ψ_r will be identical. It is interesting to note in the test problem that although each matrix in Eq. (48) is complex, the final flexibility matrix D_p is real. Therefore, the state residual elastic flexibility matrix D_h, the residual inertia-relief attachment mode matrices Φ_d and Ψ_d, and other matrices such as Φ_{bd}^D, $\bar{A}_{dd}$, $\bar{B}_{dd}$, a_ϕ, a_ψ, a, b, etc., are all real. Thus, the number of storage locations and computations may be greatly reduced.

Conclusions

A component mode synthesis procedure for structures with arbitrary linear damping has been presented. Projection matrices are defined that permit the method to be employed when

Table 2a Comparison of exact and approximate system modal damping coefficients for test problem

Mode order	Exact σ	Case 1 σ	Case 1 Error, %	Case 2 σ	Case 2 Error, %	Case 3 σ	Case 3 Error, %
1	−0.47268E−3	−0.490E−3	3.692	−0.474E−3	0.251	−0.47268E−3	0.00076
2	−0.31164E−2	−0.318E−2	2.145	−0.312E−2	0.176	−0.31165E−2	0.00266
3	−0.10885E−1	−0.110E−1	0.993	−0.109E−1	0.186	−0.10885E−1	0.00352
4	−0.29957E−1	−0.314E−1	4.712	−0.302E−1	0.943	−0.29970E−1	0.0442
5	−0.67127E−1	−0.673E−1	0.267	−0.673E−1	0.211	−0.67133E−1	0.00889
6	−0.13117E0	−0.138E0	5.307	−0.135E0	2.660	−0.13142E0	0.187
7	−0.23999E0	−0.251E0	4.464	−0.247E0	3.005	−0.24050E0	0.214
8	−0.39527E0	−0.407E0	2.885	−0.405E0	2.415	−0.39658E0	0.332
9	−0.62119E0	−0.749E0	20.64	−0.732E0	17.87	−0.63006E0	1.428
10	−0.10944E+1	−0.112E+1	2.233	−0.112E+1	2.266	−0.11062E+1	1.073

Only model damping coefficients corresponding to flexible complex modes are included in this table. Values of σ_i for cases 1 and 2 from Ref. 8 have only three significant digits. All σ_i are the real part of one of a pair of complex conjugate system eigenvalues.

Table 2b Comparison of exact and approximate system damped natural frequencies for test problem

Mode order	Exact ω_d	Case 1 ω_d	Case 1 Error, %	Case 2 ω_d	Case 2 Error, %	Case 3 ω_d	Case 3 Error, %
1	0.22374E0	0.224E0	−0.001	0.224E0	0.006	0.22374E0	0.00028
2	0.61688E0	0.617E0	0.015	0.617E0	0.021	0.61688E0	0.00122
3	0.12101E+1	0.121E+1	0.027	0.121E+1	0.039	0.12101E+1	0.00243
4	0.20032E+1	0.201E+1	0.338	0.201E+1	0.345	0.20037E+1	0.0243
5	0.30002E+1	0.300E+1	0.021	0.300E+1	0.052	0.30003E+1	0.00532
6	0.42069E+1	0.426E+1	1.226	0.426E+1	1.241	0.42113E+1	0.105
7	0.56300E+1	0.570E+1	1.248	0.570E+1	1.283	0.56361E+1	0.108
8	0.72698E+1	0.735E+1	1.102	0.735E+1	1.150	0.72835E+1	0.188
9	0.90249E+1	0.972E+1	7.674	0.972E+1	7.692	0.90886E+1	0.705
10	0.11965E+2	0.121E+2	0.928	0.121E+2	1.038	0.12035E+2	0.588

Only damped natural frequencies corresponding to flexible complex modes are included in this table. Values of ω_i for cases 1 and 2 from Ref. 8 have only three significant digits. All ω_{d_i} are the imaginary part of one of a pair of complex conjugate system eigenvalues.

components have rigid-body freedom. Residual inertia-relief attachment modes in state-variable form, which acount for the flexibility of the complex modes not retained explicitly in the model, are defined. Results of a single numerical problem indicate that the method proposed in this paper produces reduced-order models having very accurate system eigenvalues.

Although the test problem described in this paper is nonclassically damped, the mass, stiffness, and damping matrices of this problem are all symmetric. The procedure now needs to be tested on more general systems, and convergence studies need to be conducted.

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